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ON A SUPERATOMIC BOOLEAN ALGEBRA WHICH IS NOT GENERATED BY A WELL-FOUNDED SUBLATTICE

BY

URI ABRAHAM AND MATATYAHU RUBIN

Department of Mathematics, Ben Gurion University of the Negev
Beer Sheva 84105, Israel
e-mail: abraham@math.bgu.ac.il

AND

ROBERT BONNET

Laboratoire de Mathématiques, Université de Savoie
Le Bourget-du-Lac, France
e-mail: bonnet@in2p3.fr

ABSTRACT

Let $\mathfrak{b}$ denote the unboundedness number of ω^ω. That is, $\mathfrak{b}$ is the smallest cardinality of a subset $F \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is $f \in F$ such that $\{n\colon g(n) \leq f(n)\}$ is infinite. A Boolean algebra B is well-generated, if it has a well-founded sublattice L such that L generates B. We show that it is consistent with ZFC that $\aleph_1 < 2^{\aleph_0} = \mathfrak{b}$, and there is a Boolean algebra B such that B is not well-generated, and B is superatomic with cardinal sequence $\langle \aleph_0, \aleph_1, \aleph_1, 1\rangle$. This result is motivated by the fact that if the cardinal sequence of a Boolean algebra B is $\langle \aleph_0, \aleph_0, \lambda, 1\rangle$, and B is not well-generated, then $\lambda \geq \mathfrak{b}$.

1. Introduction

This work is concerned with superatomic Boolean algebras which are not well-generated. We briefly explain the relevant notions.

A Boolean algebra is **superatomic**, if for some ordinal α, the α's Cantor Bendixon derivative of B is finite (more details are given below). There exists

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a vast literature concerning superatomic Boolean algebras, and Koppelberg [Ko] can be consulted. In order to investigate superatomicity, Bonnet and Rubin introduced in [BR1] the notion of well-generatedness of Boolean algebras, which is a natural class of superatomic Boolean algebras.

A Boolean algebra B is **well-generated**, if it has a subset L such that:
(1) L is a sublattice of B, that is, L is closed under the join and meet of B;
(2) L generates B;
(3) $\langle L, \leq^B \restriction L\rangle$ is well founded.

The idea is to define a class of Boolean algebras that resembles in some way the ordinals themselves and which have, by virtue of this resemblance, a terminating Cantor Bendixon derivation. Though every well-generated Boolean algebra is superatomic, the converse is not true (see [BR1]), and the general type of question that motivates this paper is to investigate when a superatomic Boolean algebra is well-generated. Two basic examples of superatomic non-well-generated Boolean algebras are constructed in [BR1]. These are of cardinal sequences $\langle \aleph_1, \aleph_0, \aleph_1, 1\rangle$ and $\langle \aleph_0, \aleph_0, 2^{\aleph_0}, 1\rangle$. Assuming MA $\wedge$ $(\aleph_1 < 2^{\aleph_0})$, we show there that every algebra with cardinal sequences $\langle \aleph_0, 2^{\aleph_0}, \aleph_1, 1\rangle$ is well-generated.

We describe now in details the Cantor Bendixon derivation in Boolean algebras. This is the dual of the well-known derivation in Hausdorff compact 0-dimensional spaces. Let B be a Boolean algebra. $\mathrm{At}(B)$ denotes the set of atoms of B, and $I^{\mathrm{At}}(B)$ denotes the ideal of B generated by $\mathrm{At}(B)$. We define by induction on ordinals the sequence of **canonical ideals** of B. Let $I_0(B) = \{0^B\}$. Suppose that the ideal $I_\alpha(B)$ has been defined. Let $\varphi_\alpha\colon B \to B/I_\alpha(B)$ denote the canonical homomorphism from B onto $B/I_\alpha(B)$. We define

$$I_{\alpha+1}(B) = \varphi_\alpha^{-1}(I^{\mathrm{At}}(B/I_\alpha(B))).$$

If δ is a limit ordinal, then

$$I_\delta(B) = \bigcup_{\gamma<\delta} I_\gamma(B).$$

So B is superatomic if, for some α, $B/I_\alpha(B)$ is finite. The first such α is called the rank of B, and is denoted by $\mathrm{rk}(B)$. However, if $B = \{0^B\}$, then $\mathrm{rk}(B)$ is defined to be -1. Let $\lambda_\alpha(B) = |\mathrm{At}(B/I_\alpha(B))|$. The sequence $\langle \lambda_\alpha(B)\colon \alpha \leq \mathrm{rk}(B)\rangle$ is called the **cardinal sequence** of B. Note that $\lambda_\alpha(B)$ is the number of isolated points in the α's Cantor Bendixon's derivative of the Stone space of B.

In [BR1] we investigate extensively the well-generatedness of Boolean algebras with finite and countable ranks. We particularly concentrate on subalgebras of

$\wp(\omega)$. Two facts there (Corollary 3.11 and Theorem 3.4(a)) motivate the present work.

Let $\mathfrak{b}$ denote the unboundedness number of ω^ω. That is, $\mathfrak{b}$ is the smallest cardinality of a subset $F \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is $f \in F$ such that $\{n\colon g(n) \leq f(n)\}$ is infinite.

FACT 1: If B is a Boolean algebra with cardinal sequence $\langle \aleph_0, \aleph_0, \lambda, 1\rangle$ which is not well-generated, then $\lambda \geq \mathfrak{b}$.

FACT 2: There is a Boolean algebra B with cardinal sequence $\langle \aleph_0, \aleph_0, \mathfrak{b}, 1\rangle$ which is not well-generated.

Facts 1 and 2 raise the question whether every non-well-generated subalgebra B of $\wp(\omega)$ with finite rank uses in some way an unbounded family of functions from ω^ω, in which case the cardinal sequence of every such B should include a cardinal $\geq \mathfrak{b}$.

In this work we show that it is consistent that the answer is negative. In fact, we prove the following theorem.

THEOREM 1.1: *It is consistent with ZFC that $\aleph_1 < 2^{\aleph_0} = \mathfrak{b}$, and there is a non-well-generated Boolean algebra with cardinal sequence $\langle \aleph_0, \aleph_1, \aleph_1, 1\rangle$.*

Also in [BR1] (Corollary 4.2), we show the following relevant fact.

FACT 3: If MA holds and $\aleph_1 < 2^{\aleph_0}$, then every Boolean algebra with cardinal sequence $\langle \aleph_0, 2^{\aleph_0}, \aleph_1, 1\rangle$ is well-generated (and it follows that every Boolean algebra with cardinal sequence $\langle \aleph_0, \aleph_1, \aleph_1, 1\rangle$ is well-generated).

So the result proved to be consistent in Theorem 1.1, cannot be proved in ZFC. We ask what can be proved in ZFC, namely for which cardinal sequences are there superatomic non-well-generated Boolean algebras with these cardinal sequences? The first type of Boolean algebras that comes to mind in this context is the family of subalgebras of $\wp(\omega)$. The results of [BR1] lead us to consider such Boolean algebras of rank 3, and to pose the following problem.

PROBLEM 1: *Does* ZFC *imply that if $\aleph_2 < 2^{\aleph_0}$, then there is a cardinal λ and a Boolean algebra B with cardinal sequence $\langle \aleph_0, \lambda, \aleph_2, 1\rangle$, such that B is not well-generated?*

That is, for some λ the cardinal sequence $\langle \aleph_0, \lambda, \aleph_2, 1\rangle$ admits a non-well-generated Boolean algebra regardless of whether or not $\aleph_2 \geq \mathfrak{b}$, as long as $\aleph_2 \leq 2^{\aleph_0}$. We do not know to prove or disprove the above statement. We also do not know the answer when λ is fixed to be $\aleph_1$, or when λ is fixed to be $\aleph_2$.

Talayco in [Ta] defined the following property (which we call "stationary Knaster"): We say that a poset $\langle P, <\rangle$ has the **stationary Knaster property**, if for every stationary set $u \subseteq \aleph_1$ and a sequence of members of P, $\langle p_\alpha\colon \alpha \in u\rangle$, there is a stationary set $v \subseteq u$ such that for every $\alpha, \beta \in v$, p_α and p_β are compatible. In [Ta], Blass proved that this property is preserved in finite support iteration. A proof of this theorem is given here for completness.

For general facts on Boolean algebras, see Koppelberg [Ko]. The notion of well-generated Boolean algebras is presented and dealt with in [BR1] and [BR2]. General facts in set theory and forcing can be found in Jech [Je]. For an example of the use of the Knaster property in forcing, see Kunen and Tall [KT].

2. Tail systems

For a set of ordinals u let $u^{\lim}$ denote the set of all ordinals $< \sup(u)$ which are accumulation points of u. So $\aleph_1^{\lim}$ denotes the set of countable limit ordinals. For sets a, b let $a \precsim^{\text{fin}} b$ denote that $a - b$ is finite, $a \sim^{\text{fin}} b$ denote that $a - b$ and $b - a$ are finite, and $a \perp^{\text{fin}} b$ denote that $a \cap b$ is finite. Let $\wp(A)$ denote the powerset of A.

Arrowed letters like $\vec{a}$ denote sequences, that is, functions whose domain is a set of ordinals. If $\vec{a}$ is a sequence, we usually denote the elements of $\vec{a}$ by a_α. However, for sequences $\vec{s}$ defined on ω, the nth element is denoted $s(n)$. So $\vec{a}$ usually stands for $\langle a_\alpha\colon \alpha \in \operatorname{Dom}(\vec{a})\rangle$. We sometimes use the notation $\vec{a} = \{a_\alpha\colon \alpha \in J\}$ in two meanings. This denotes the obvious sequence, and it also denotes its range.

Definition 2.1: (a) A **tail system** is a pair $\langle \vec{a}, \vec{b}\rangle$ with

$$\vec{a} = \langle a_\alpha\colon \alpha \in \aleph_1\rangle \quad \text{and} \quad \vec{b} = \langle b_\delta\colon \delta \in \aleph_1^{\lim}\rangle,$$

where $a_\alpha, b_\delta \subseteq \omega$ for every $\alpha \in \aleph_1$ and $\delta \in \aleph_1^{\lim}$, and the following holds.

(1) $\vec{a}$ is an almost disjoint family consisting of infinite subsets of ω. That is, for every distinct α and β, $a_\alpha \subseteq \omega$ is infinite, and $a_\alpha \perp^{\text{fin}} a_\beta$.

(2) For every $\delta \in \aleph_1^{\lim}$, there is a strictly increasing sequence $\vec{s}^{\,\delta} = \{s^\delta(n)\colon n \in \omega\}$ converging to δ such that for every $n \in \omega$, $a_{s^\delta(n)} \precsim^{\text{fin}} b_\delta$, and for every $\alpha \in \aleph_1 - \operatorname{Rng}(\vec{s}^{\,\delta})$, $a_\alpha \perp^{\text{fin}} b_\delta$. So $\vec{s}^{\,\delta}$ can be defined from b_δ and $\{a_\alpha\colon \alpha \in \delta\}$ as the sequence of those a_α for which $a_\alpha \precsim^{\text{fin}} b_\delta$.

(3) For distinct $\delta, \delta' \in \aleph_1^{\lim}$ let $J_{\delta,\delta'} = \operatorname{Rng}(\vec{s}^{\,\delta}) \cap \operatorname{Rng}(\vec{s}^{\,\delta'})$. So $J_{\delta,\delta'}$ is finite. For every distinct $\delta, \delta' \in \aleph_1^{\lim}$,

$$(b_\delta - \bigcup\{a_\alpha\colon \alpha \in J_{\delta,\delta'}\}) \perp^{\text{fin}} (b_{\delta'} - \bigcup\{a_\alpha\colon \alpha \in J_{\delta,\delta'}\}).$$

(b) Let $\beta \leq \aleph_1$ be an ordinal, and $\vec{a} = \{a_\alpha : \alpha \in \beta\} \subseteq \wp(\omega)$ be a sequence of sets. Let $u \subseteq \beta$ and $\vec{a}' = \{a'_\alpha : \alpha \in u\}$. We say that $\vec{a}'$ is **near** to $\vec{a}$, if for every $\alpha \in u$, $a'_\alpha \sim^{\text{fin}} a_\alpha$.

We are interested in tail systems that have certain guessing properties, and we give now four definitions of variants of these guessing properties.

Definition 2.2: (a) *Special tail systems.* A tail system $\langle \vec{a}, \vec{b} \rangle$ is special if the following holds. For every uncountable set $u \subseteq \aleph_1$, and a family $\vec{a}' = \{a'_\alpha : \alpha \in u\}$ that is near to $\vec{a}$, there is a club set of $\delta \in u^{\lim}$ for which

(i) $\operatorname{Rng}(\vec{s}^\delta) \cap u$ is infinite;
(ii) for every finite set $\sigma \subseteq \{a_\alpha : \alpha \in \aleph_1\} \cup \{b_\gamma : \gamma \in \aleph_1^{\lim}\} \cup \{\{n\} : n \in \omega\}$, $\{n \in \omega : s^\delta(n) \in u \text{ and } a'_{s^\delta(n)} \subseteq \bigcup \sigma\}$ is finite.

We make the following remarks concerning this definition. Speciality of $\langle \vec{a}, \vec{b} \rangle$ can be viewed as a combination of two properties: (1) the property of the sequences $\langle \vec{s}^\delta \mid \delta \in \aleph_1^{\lim} \rangle$ that $\vec{s}^\delta \cap u$ is infinite for every uncountable $u \subseteq \omega_1$ on a club set of δ's, and (2) a "non-inclusion" property. Observe the contrast between the relation $a'_{s^\delta(n)} \precsim^{\text{fin}} b_\delta$, which holds by definition of a tail system, and requirement (ii).

(b) *Strongly stationarily special systems.* A tail system $\langle \vec{a}, \vec{b} \rangle$ is strongly stationarily special if the following holds. For every stationary set $u \subseteq \aleph_1$ and a family $\vec{a}' = \{a'_\alpha : \alpha \in u\}$ that is near to $\vec{a}$, there is $\delta \in u^{\lim} \cap u$ such that:

(i) $\operatorname{Rng}(\vec{s}^\delta) \cap u$ is infinite;
(ii) for every finite set $\sigma \subseteq \{a_\alpha : \alpha \in \aleph_1\} \cup \{b_\gamma : \gamma \in \aleph_1^{\lim}\} \cup \{\{n\} : n \in \omega\}$, $\{n \in \omega : s^\delta(n) \in u \text{ and } a'_{s^\delta(n)} \subseteq \bigcup \sigma\}$ is finite.

It should be obvious that any special tail system is also strongly stationarily special.

(c) *Stationarily special systems.* A tail system $\langle \vec{a}, \vec{b} \rangle$ is stationarily special if the following holds. For every stationary set $u \subseteq \aleph_1$ and a family $\vec{a}' = \{a'_\alpha : \alpha \in u\}$ that is near to $\vec{a}$, there is a stationary set $T \subseteq u$ such that for every $\delta \in T$, δ is a limit ordinal and the following holds. For every finite set $\sigma \subseteq \{a_\alpha : \alpha \in \aleph_1\} \cup \{b_\gamma : \gamma \in \aleph_1^{\lim}\} \cup \{\{n\} : n \in \omega\}$,

$$\{n \in \omega : s^\delta(n) \in u \text{ and } a'_{s^\delta(n)} \not\subseteq \bigcup \sigma\} \text{ is infinite.}$$

We prove below that an equivalent formulation $(*)$ of this property of $\langle \vec{a}, \vec{b} \rangle$ is obtained by replacing the quantification:

"there is a stationary set $T \subseteq u$ such that for every $\delta \in T$ the following holds" with the quantification:

"there is some $\delta \in u$ such that the following holds".

(d) *Weakly special systems.* A tail system $\langle \vec{a}, \vec{b} \rangle$ is weakly special if the following holds. For every $\vec{a}' = \{a'_\alpha: \alpha \in \aleph_1\}$ that is near to $\vec{a}$, there is $\delta \in \aleph_1^{\lim}$ such that: for every finite set σ, if

$$\sigma \subseteq \{\{n\}: n \in \omega\} \cup \{a_\alpha: \alpha \in \aleph_1\} \cup \{b_\delta: \delta \in \aleph_1^{\lim}\},$$

then the set $\{n \in \omega: a'_{s^\delta(n)} \not\subseteq \bigcup \sigma\}$ is infinite.

We check that the variant (∗) of specialness is indeed equivalent to stationary specialness. Obviously stationary specialness implies (∗).

Suppose that $\langle \vec{a}, \vec{b} \rangle$ satisfies (∗), and assume by contradiction that the set of "good" δ's is not stationary. That is, there is a club C such that for every limit $\delta \in C$, there is a finite set $\sigma \subseteq \{a_\alpha: \alpha \in \aleph_1\} \cup \{b_\gamma: \gamma \in \aleph_1^{\lim}\} \cup \{\{n\}: n \in \omega\}$, such that $\{n \in \omega: s^\delta(n) \in u$ and $a'_{s^\delta(n)} \not\subseteq \bigcup \sigma\}$ is finite. Let $\vec{a}'' = \vec{a}' \restriction (u \cap C)$. Obviously $\vec{a}''$ does not have any good δ. So (∗) does not hold. This is a contradiction.

The following implications between the notions of "specialness" hold:

(†) special $\Rightarrow$ strongly stationarily special $\Rightarrow$ stationarily special $\Rightarrow$ weakly special.

We have shown that a special tail system is strongly stationarily special. If the variant (∗) which is equivalent to the notion of stationary specialness is considered, then the second implication, from strongly to plain stationarily special, is also clear. It is also obvious that a stationarily special system is weakly special.

Let $\diamondsuit^*_{\aleph_1}$ denote the following axiom due to Jensen [Jn].

There is a sequence $\vec{S} = \{S_\alpha: \alpha < \aleph_1\}$ (called a "diamond star" sequence) such that:

(i) for every $\alpha \in \aleph_1$, $S_\alpha \subseteq \wp(\alpha)$ and $|S_\alpha| \leq \aleph_0$;

(ii) for every stationary set $u \subseteq \aleph_1$ and a subset $A \subseteq \aleph_1$ there is $\alpha \in u$ such that $A \cap \alpha \in S_\alpha$.

(Note that this just means that the set of ordinals on which A is guessed contains a closed and unbounded set.)

THEOREM 2.3: *Assuming $\diamondsuit^*_{\aleph_1}$ there is a special tail system.*

Proof: First we state without proof a very simple lemma.

LEMMA 2.4: *Let Y be a countable collection of subsets of ω such that $Y = Y_0 \cup Y_1$, and the following holds. (1) For every finite $F \subseteq Y_1$, $\omega \setminus \bigcup F$ is infinite. (2) For* *every* $e_0 \in Y_0$ *and* $e_1 \in Y_1$, $e_0 \perp^{\text{fin}} e_1$. *Then there exists an infinite* $x \subseteq \omega$ *such that* $e \precsim^{\text{fin}} x$ *for every* $e \in Y_0$, *and* $e \perp^{\text{fin}} x$ *for every* $e \in Y_1$.

(Remark that the lemma holds even when Y_0 is empty.)

The construction of the special tail system is carried inductively. For any countable ordinal ε and sequences $\vec{a} = \{a_\alpha : \alpha \in \varepsilon\}$ and $\vec{b} = \{b_\delta : \delta \in \varepsilon^{\text{lim}}\}$, we say that $\langle \vec{a}, \vec{b} \rangle$ is a short tail system of length ε, if it satisfies clauses (1), (2) and (3) of Definition 2.1 formulated for $\alpha, \beta, \delta, \delta' < \varepsilon$.

CLAIM 2.5: (a) *Suppose that* $\varepsilon \in \aleph_1^{\text{lim}}$ *and* $\langle \vec{a}, \vec{b} \rangle$ *is a short tail system of length* ε. *We denote* $A = \{a_\alpha : \alpha \in \varepsilon\}$ *and* $B = \{b_\delta : \delta \in \varepsilon^{\text{lim}}\}$. *Let* $C \subseteq \wp(\omega)$ *be a countably infinite collection of infinite sets such that for every distinct* $c, c' \in C$, $c \perp^{\text{fin}} c'$, *and for every* $c \in C$ *and* $e \in A \cup B$, $c \perp^{\text{fin}} e$. *Let* $\vec{s} = \langle s(n) : n \in \omega \rangle$ *be a strictly increasing sequence of ordinals converging to* ε. *Then there are* a_ε, *and* b_ε *such that:*

(i) $\langle \{a_\alpha : \alpha \in \varepsilon + 1\}, \{b_\delta : \delta \in (\varepsilon+1)^{\text{lim}}\} \rangle$ *is a short tail system of length* $\varepsilon + 1$;

(ii) $\vec{s}^{\,\varepsilon} = \vec{s}$ *(which means that* $a_\alpha \precsim^{\text{fin}} b_\varepsilon$ *for* $\alpha < \varepsilon$ *iff* α *is on* $\vec{s}$*)*;

(iii) *for every* $c \in C$, $a_\varepsilon \perp^{\text{fin}} c$ *and* $b_\varepsilon \perp^{\text{fin}} c$.

(b) Suppose that $\varepsilon \in \aleph_1$ *is a successor ordinal. Let* $\langle \vec{a}, \vec{b} \rangle$, A, B *and* C *be as in Part* (a). *Then there is* a_ε *such that* (i) $\langle \{a_\alpha : \alpha \in \varepsilon + 1\}, \{b_\delta : \delta \in (\varepsilon+1)^{\text{lim}}\} \rangle$ *is a short tail system of length* $\varepsilon + 1$ *and* (ii) *for every* $c \in C$, $a_\varepsilon \perp^{\text{fin}} c$.

Proof: The proof of Part (b) is very simple, so we only prove Part (a). For every $\delta \in \varepsilon^{\text{lim}}$ define

$$b'_\delta = b_\delta \setminus \bigcup \{a_\alpha : \alpha \in \operatorname{Rng}(\vec{s}^{\,\delta}) \cap \operatorname{Rng}(\vec{s})\}.$$

Namely, b'_δ is obtained from b_δ be removing those (finitely many) a_α's that are listed both in the ω-sequence $\vec{s}^{\,\delta}$ converging to δ and the given ω-sequence converging to ε. Now the collections

$$Y_0 = \{a_\alpha : \alpha \in \operatorname{Rng}(\vec{s})\} \quad \text{and} \quad Y_1 = (A \setminus Y_0) \cup \{b'_\delta : \delta \in \varepsilon^{\text{lim}}\} \cup C$$

satisfy the lemma's conditions (since C is infinite no finite subset of Y_1 covers ω). Hence there is b_ε such that $e \precsim^{\text{fin}} b_\varepsilon$ for $e \in Y_0$, and $e \perp^{\text{fin}} b_\varepsilon$ for every $e \in Y_1$.

No finite subset of $A \cup B \cup C \cup \{b_\varepsilon\}$ covers ω and hence an infinite a_ε can be defined which is almost disjoint to each of these sets. This completes the proof of Claim 2.5.

We now turn to the inductive construction of the tail sequence. Let $\vec{S} = \langle S_\alpha : \alpha \in \aleph_1 \rangle$ be a $\diamondsuit^*_{\aleph_1}$-sequence. We may assume that for every $\alpha \in \aleph_1$, $S_\alpha \subseteq$

$\wp(\alpha \times \alpha)$ is countable, and that $\vec{S}$ guesses subsets of $\aleph_1 \times \aleph_1$ rather than subsets of $\aleph_1$. For $u \subseteq \aleph_1$ and a u-sequence of subsets of ω, $\vec{e} = \{e_\alpha : \alpha \in u\}$, we define $S(\vec{e}) = \{\langle \alpha, k\rangle : \alpha \in u \text{ and } k \in e_\alpha\}$. Hence $S(\vec{e}) \subseteq \aleph_1 \times \aleph_1$. For $S \subseteq \aleph_1 \times \aleph_1$ let $u^S = \{\alpha : \exists \beta(\langle \alpha, \beta\rangle \in S)\}$, and for $\alpha \in u^S$ let $a^S_\alpha = \{\beta : \langle \alpha, \beta \rangle \in S\}$, and let $\vec{a}^S = \{a^S_\alpha : \alpha \in u^S\}$.

We define by induction on $\varepsilon < \aleph_1$ a short tail system $\langle \vec{a}^{\,\varepsilon}, \vec{b}^{\,\varepsilon}\rangle$ of height ε, and a countable, auxiliary collection $C^{\,\varepsilon}$ of infinite subsets of ω, such that:

1. $\vec{a}^{\,\varepsilon} = \langle a_\alpha : \alpha \in \varepsilon\rangle$, $\vec{b}^{\,\varepsilon} = \langle b_\delta : \delta \in \varepsilon^{\lim}\rangle$. Finally, the pair $(\langle a_\alpha : \alpha \in \aleph_1\rangle, \langle b_\delta : \delta \in \aleph_1\rangle)$ will be the required tail system.
2. $a_\alpha \perp^{\text{fin}} c$ and $b_\delta \perp^{\text{fin}} c$ for every $c \in C^\varepsilon$, $\alpha < \varepsilon$ and $\delta \in \varepsilon^{\lim}$.
3. If $\varepsilon_1 < \varepsilon_2$ then $C^{\varepsilon_1} \subseteq C^{\varepsilon_2}$. Similarly, the short tail system at ε_1 is extended by that at ε_2. This justifies our notation, e.g., a_α (rather than a^ε_α) for the α's member of the sequence $\vec{a}^{\,\varepsilon}$.

We start with $\{a_0\} \cup C^0$ being any countable, infinite, pairwise disjoint collection of infinite subsets of ω.

If ε is a limit ordinal, and $\vec{a}^{\,\nu}$, $\vec{b}^{\,\nu}$ and $C^{\,\nu}$ were defined for every $\nu < \varepsilon$, then let $\vec{a}^{\,\varepsilon} = \bigcup_{\nu<\varepsilon} \vec{a}^{\,\nu}$, $\vec{b}^{\,\varepsilon} = \bigcup_{\nu<\varepsilon} \vec{b}^{\,\nu}$, and $C^{\,\varepsilon} = \bigcup_{\nu<\varepsilon} C^{\,\nu}$.

Suppose that $\vec{a}^{\,\varepsilon}$, $\vec{b}^{\,\varepsilon}$ and $C^{\,\varepsilon}$ were defined.

CASE 1: ε is a successor ordinal. Let a_ε be obtained by Part (b) of Claim 2.5. We define $\vec{a}^{\,\varepsilon+1}$ and $\vec{b}^{\,\varepsilon+1}$ in the obvious way, and $C^{\,\varepsilon+1} = C^{\,\varepsilon}$.

CASE 2: ε is a limit ordinal. Define

$$S'_\varepsilon = \{T \in S_\varepsilon : u^T \text{ is an unbounded subset of } \varepsilon, \text{ and } \vec{a}^{\,T} \text{ is a sequence of subsets of } \omega \text{ which is near to } \vec{a}^{\,\varepsilon}\}.$$

Let $\vec{s} \stackrel{\text{def}}{=} \vec{s}_\varepsilon$ be a strictly increasing ω-sequence converging to ε such that for every $T \in S'_\varepsilon$, $\operatorname{Rng}(\vec{s}) \cap u^T$ is infinite. Such a sequence exists, since $|S'_\varepsilon| \leq \aleph_0$, and for every $T \in S'_\varepsilon$, u^T is unbounded in ε. We denote $\vec{s}(n)$ by $s(n)$.

We now define a sequence of subsets of ω, $\vec{a}'_\varepsilon = \{a'_{\varepsilon,\alpha} : \alpha \in \operatorname{Rng}(\vec{s})\}$ that is near to $\vec{a}^{\,\varepsilon}$. Let $\{\vec{e}_{\varepsilon,i} : i \in \omega\}$ be an enumeration of $\{\vec{a}^{\,T} : T \in S'_\varepsilon\}$ (if it is non-empty). So $\vec{e}_{\varepsilon,i}$ is a sequence of subsets of ω which is near to $\vec{a}^{\varepsilon}$, and $\vec{e}_{\varepsilon,i}(\alpha)$ denotes the α's member of this sequence. For every $n \in \omega$ let

$$a'_{\varepsilon,s(n)} = a_{s(n)} \cap \bigcap \{\vec{e}_{\varepsilon,i}(s(n)) : i < n \text{ and } s(n) \in \operatorname{Dom}(\vec{e}_{\varepsilon,i})\}.$$

Since $\vec{e}_{\varepsilon,i}$ is near to $\vec{a}^{\,\varepsilon}$, $\vec{e}_{\varepsilon,i}(s(n)) \sim^{\text{fin}} a_{s(n)}$, and it follows that $a'_{\varepsilon,s(n)} \sim^{\text{fin}} a_{s(n)}$. (It may be that the set whose intersection is taken is empty, but in this case $a'_{\varepsilon,s(n)} = a_{s(n)} \cap \bigcap \emptyset = a_{s(n)}$.)

So $\mathrm{Dom}\,(\vec{a}'_\varepsilon) = \mathrm{Rng}\,(\vec{s}_\varepsilon)$, and $\vec{a}'_\varepsilon$ is near to $\vec{a}^{\,\varepsilon}$.

Now define a strictly increasing sequence $\{n_i \in \omega\colon i \in \omega\}$ by the requirements (1) that $n_i \in a'_{\varepsilon,s(i)}$, and (2) that the set $c_\varepsilon = \{n_i\colon i \in \omega\}$ is almost disjoint to any set in $A^\varepsilon \cup B^\varepsilon \cup C^\varepsilon$. (Use here the fact that if $\delta < \alpha$ then $b_\delta \perp^{\mathrm{fin}} a_\alpha$.) Define $C^{\varepsilon+1} = C^\varepsilon \cup \{c_\varepsilon\}$. In our proof below we are going to use the following obvious property of c_ε: If x is such that $a'_{\varepsilon,\alpha} \subseteq x$ for infinitely many α's in s, then $x \cap c_\varepsilon$ is infinite.

Now apply Claim 2.5(a) to $\langle \vec{a}^{\,\varepsilon}, \vec{b}^{\,\varepsilon}\rangle$, and $C^{\varepsilon+1}$, and obtain a_ε and b_ε. We define $\vec{a}^{\,\varepsilon+1}$, and $\vec{b}^{\,\varepsilon+1}$ in the obvious way. It is easy to check that the induction hypotheses hold.

Let $\vec{a} = \langle a_\alpha\colon \alpha \in \aleph_1\rangle$ and $\vec{b} = \langle b_\delta\colon \delta \in \aleph_1^{\mathrm{lim}}\rangle$. We show that $\langle \vec{a}, \vec{b}\rangle$ is a special tail system. Write $C = \bigcup_{\alpha<\aleph_1} C^\alpha$.

It is clear from the construction that $\langle \vec{a}, \vec{b}\rangle$ is a tail system, and that:

(1) for every $\delta \in \aleph_1^{\mathrm{lim}}$, $\vec{s}^{\,\delta} = \vec{s}_\delta$,

(2) for every $e \in \mathrm{Rng}\,(\vec{a}) \cup \mathrm{Rng}\,(\vec{b})$ and $c \in C$, $e \perp^{\mathrm{fin}} c$.

We must prove that $\langle \vec{a}, \vec{b}\rangle$ is special. Suppose that $u \subseteq \aleph_1$ is uncountable, $\vec{a}' = \{a'_\alpha\colon \alpha \in u\}$ is a sequence near to $\vec{a}$. Apply the guessing property of the diamond sequence to $S(\vec{a}^{\,'})$ and obtain a closed unbounded set $D \subseteq u^{\mathrm{lim}}$ such that for every $\delta \in D$

$$T \overset{\mathrm{def}}{=} S(\vec{a}^{\,'}) \cap (\delta \times \delta) \in S_\delta.$$

Clearly $u^T = u \cap \delta$ is unbounded in δ. Also, $\vec{a}^{\,T} = \vec{a}'\restriction\delta$. Hence $\vec{a}^{\,T}$ is near to $\vec{a}^{\,\delta}$, and so

$$T \in S'_\delta.$$

It follows from this that $\mathrm{Rng}\,(\vec{s}^{\,\delta}) \cap u$ is infinite, and hence δ satisfies Clause (i) in the definition of a special tail system.

We next see that δ satisfies clause (ii) in that definition. Let $\sigma \subseteq \mathrm{Rng}\,(\vec{a}) \cup \mathrm{Rng}\,(\vec{b}) \cup \{\{n\}\colon n \in \omega\}$ be finite. We show that

$$E \overset{\mathrm{def}}{=} \{n \in \omega\colon s^\delta(n) \in u \text{ and } a'_{s^\delta(n)} \subseteq \bigcup \sigma\} \text{ is finite.}$$

We denote $\vec{s}^{\,\delta}$ by $\vec{s}$ and $s^\delta(n)$ by $s(n)$. Consider the set c_δ obtained at stage δ of the construction. Since each set in σ is almost disjoint to c_δ, $\cup\delta \perp^{\mathrm{fin}} c_\delta$ and there is m such that $(\bigcup\sigma) \cap c_\delta \setminus m = \emptyset$. Recall that $c_\delta = \{n_i\colon i \in \omega\}$ and $n_i \in a'_{\delta,s(i)}$. Hence, for every i such that $n_i \geq m$, $n_i \notin \bigcup\sigma$ and thus

$$a'_{\delta,s(i)} \not\subseteq \bigcup \sigma.$$

Let k be such that $T = \vec{e}_{\delta,k}$. Now for every $n > k, m$ such that $s(n) \in \mathrm{Dom}\,(\vec{a}^{\,T})$, $a'_{\delta,s(n)} \subseteq \vec{a}^{\,T}(s(n)) = a'_{s(n)}$, and hence $a'_{s(n)} \not\subseteq \bigcup\sigma$. ∎

3. On tail-system algebras

The following definition shows how tail systems are used in this paper to produce Boolean algebras.

Definition 3.1: Let $\langle \vec{a}, \vec{b} \rangle$ be a tail system. The subalgebra B of $\wp(\omega)$ generated by $\{\{n\}: n \in \omega\} \cup \{a_\alpha: \alpha \in \aleph_1\} \cup \{b_\delta: \delta \in \aleph_1^{\lim}\}$ is called the **tail system algebra** of $\langle \vec{a}, \vec{b} \rangle$. We say that B is (stationarily, weakly) special, if $\langle \vec{a}, \vec{b} \rangle$ is (stationarily, weakly) special.

We introduce some notations. Let B be a superatomic Boolean algebra. We say that B is **unitary**, if $\lambda_{\mathrm{rk}\,(B)}(B) = 1$. We denote $I_{\mathrm{rk}\,(B)}(B)$ by $I(B)$. Let

$$\widehat{\mathrm{At}}_\alpha(B) \stackrel{\mathrm{def}}{=} \{a \in B: a/I_\alpha(B) \in \mathrm{At}\,(B/I_\alpha(B))\}.$$

Let

$$\widehat{\mathrm{At}}(B) \stackrel{\mathrm{def}}{=} \bigcup \{\widehat{\mathrm{At}}_\alpha(B): \alpha \leq \mathrm{rk}\,(B)\}.$$

A set $C \subseteq \widehat{\mathrm{At}}_\alpha(B)$ is a **complete set of representatives** for $\widehat{\mathrm{At}}_\alpha(B)$, if for every $a \in \widehat{\mathrm{At}}_\alpha(B)$ there is a unique $c \in C$ such that $c/I_\alpha(B) = a/I_\alpha(B)$. Let $a \in B$. $B \restriction a$ denotes the Boolean algebra induced by B on the set $\{c \in B: c \leq a\}$. If $a \in B - \{0^B\}$, then $\mathrm{rk}^B(a) \stackrel{\mathrm{def}}{=} \mathrm{rk}\,(B \restriction a)$.

Let $a, b \in B$. Then $a \preceq^B b$ means that $\mathrm{rk}^B(a - b) < \mathrm{rk}^B(a)$ or $a = 0^B$. Also, $a \sim^B b$ means that $a \preceq^B b$ and $b \preceq^B a$. Note that $\preceq^B$ is not a transitive relation. But the following is trivially true.

(1) If $a \preceq^B b \leq c$, then $a \preceq^B c$.

The following facts follow easily from the definitions.

(2) If $a \in \widehat{\mathrm{At}}(B)$ and $a = a_1 + \cdots + a_n$, then for some $i \leq n$, $a_i \sim^B a$.

(3) If $a \preceq^B c$ and $a' \sim^B a$, then $a' \preceq^B c$.

Let B be a Boolean algebra and $C \subseteq B$. We denote by $\mathrm{cl}_B^{\mathrm{lat}}(C)$ the lattice closure of C in B. That is, $\mathrm{cl}_B^{\mathrm{lat}}(C)$ is the closure of C under meet and join.

Before showing that a weakly special tail system algebra is not well-generated we quote two facts from [BR1].

PROPOSITION 3.2: (a) *Let B be a Boolean algebra, G a well-founded sublattice of B and C a complete set of representatives for $\widehat{\mathrm{At}}_1(B)$. Then $\mathrm{cl}_B^{\mathrm{lat}}(G \cup C)$ is well-founded.*

(b) *Let B be a well-generated algebra and I a maximal ideal in B. Then there is a well-founded sublattice $G \subseteq I$ which generates B. In particular, if B is unitary, then B has a well-generating sublattice G contained in $I(B)$.*

Proof: The proof of (a) is very simple and left to the reader (see [BR1] Proposition 3.3(a)). An outline of the proof of (b) is quoted for complexness from [BR1] Proposition 2.9(c). Let B be a well-generated algebra and I a maximal ideal of B. Let G' be a well-founded sublattice that generates B. Since I is maximal, if $a \cdot b \in I$, then $a \in I$ or $b \in I$. It follows that $G' - I$ is a sublattice of G'. As G' is a well-founded lattice, it follows that $G' - I$ has a minimum a. Let $G_1 = \{b - a\colon b \in G' - I\}$. G_1 is a sublattice of B, and since $G_1 \subseteq \mathrm{cl}^{\mathrm{lat}}_B(G' \cup \{-a\})$, it is well-founded. Let $G = \mathrm{cl}^{\mathrm{lat}}_B(G_1 \cup (G' \cap I) \cup \{-a\})$. We prove that G is as required. First $G \subseteq I$. Also $G' \subseteq \mathrm{cl}_B(G)$, and thus $\mathrm{cl}_B(G) = B$. To see that G is well-founded, observe that G_1, $G' \cap I$ and $\{-a\}$ are well-founded lattices.

In the following theorem, Part (c) is the main claim.

THEOREM 3.3: (a) *Let $\langle \vec{a},\ \vec{b}\rangle$ be a tail system and B its algebra. Then B is unitary of rank 3, $\{a_\alpha\colon \alpha \in \aleph_1\}$ is a complete set of representatives for $\widehat{\mathrm{At}}_1(B)$ and $\{b_\delta\colon \delta \in \aleph_1^{\mathrm{lim}}\}$ is a complete set of representatives for $\widehat{\mathrm{At}}_2(B)$. That is, the cardinal sequence of B is $\langle \aleph_0, \aleph_1, \aleph_1, 1\rangle$.*

(b) *Let $\langle \vec{a},\ \vec{b}\rangle$ be a tail system and B its algebra. Then for every $\alpha \in \aleph_1$ and $c \in B$, $a_\alpha \preceq^B c$ iff $a_\alpha \precsim^{\mathrm{fin}} c$, and $a_\alpha \sim^B c$ iff $a_\alpha \sim^{\mathrm{fin}} c$.*

For every $\delta \in \aleph_1^{\mathrm{lim}}$ and $c \in B$, $b_\delta \preceq^B c$ iff there is a finite set $\sigma \subseteq \{a_\alpha\colon \alpha \in \aleph_1\} \cup \{\{n\}\colon n \in \omega\}$ such that $b_\delta - c \subseteq \bigcup \sigma$.

The ideal $I(B)$ is generated by

$$\{\{n\}\colon n \in \omega\} \cup \{a_\alpha\colon \alpha \in \aleph_1\} \cup \{b_\delta\colon \delta \in \aleph_1^{\mathrm{lim}}\}.$$

(c) *If B is the algebra of a weakly special tail system, then B is not well-generated.*

Proof: (a) The verification of Part (a) is easy, but it requires some Boolean algebraic computation. Part (b) is a trivial corollary of Part (a).

(c) Let B be the algebra of the weakly special tail system $\langle \vec{a},\ \vec{b}\rangle$. Suppose by contradiction that B is well-generated. By Proposition 3.2(b), there is well-founded lattice $G \subseteq I(B)$ such that G generates B. By Proposition 3.2(a), $\mathrm{cl}^{\mathrm{lat}}_B(G \cup \{a_\alpha\colon \alpha \in \aleph_1\})$ is well-founded, so we may assume that $G \supseteq \{a_\alpha\colon \alpha \in \aleph_1\}$.

We show that for every $a \in \widehat{\mathrm{At}}(B) \cap I(B)$ there is $b \in G$ such that $a \precsim^B b$. Since G generates B, a can be written in the form $a = a_1 + \cdots + a_n$, where each a_i is the finite meet of members of G and complements of members of G. So there is $i \leq n$ such that $a_i \sim^B a$. Let $a_i = \prod \sigma_1 \cdot \prod\{-b\colon b \in \sigma_2\}$, where $\sigma_1, \sigma_2 \subseteq G$. $\sigma_1 \neq \emptyset$, since if this is not the case, then $a_i = \prod\{-b\colon b \in \sigma_2\} \notin I(B)$. This contradicts the fact that $a_i \leq a \in I(B)$. Let $b \in \sigma_1$. Then $a \preceq^B a_i \leq b$. So $a \preceq^B b$. Also $b \in G$.

For every $\delta \in \aleph_1^{\lim}$ we now choose $c_\delta \in G$ such that $b_\delta \precsim^B c_\delta$.

For every $\alpha \in \aleph_1$, $G_\alpha \stackrel{\text{def}}{=} \{b \in G\colon b \sim^B a_\alpha\}$ is a sublattice of G containing a_α. Since G_α is a nonempty sublattice of the well-founded lattice G, G_α has a minimum which we denote by a'_α. So $\vec{a}' \stackrel{\text{def}}{=} \{a'_\alpha\colon \alpha \in \aleph_1\}$ is near to $\vec{a}$.

We apply the definition of a weak specialness (Definition 2.1(d)) to $\langle \vec{a}, \vec{b}\rangle$ and to $\vec{a}'$. So there is $\delta \in \aleph_1^{\lim}$ such that

(*) for every finite set $\sigma \subseteq \{\{n\}\colon n \in \omega\} \cup \{a_\alpha\colon \alpha \in \aleph_1\} \cup \{b_\delta\colon \delta \in \aleph_1^{\lim}\}$, $\{n \in \omega\colon a'_{s^\delta(n)} \not\subseteq \bigcup \sigma\}$ is infinite.

Since $b_\delta \preceq^B c_\delta$, $F \stackrel{\text{def}}{=} \{n \in \omega\colon a_{s^\delta(n)} \not\preceq^B c_\delta\}$ is finite. We explain this. There is a finite set $\eta \subseteq \{a_\alpha\colon \alpha \in \aleph_1\} \cup \{\{n\}\colon n \in \omega\}$ such that $b_\delta - c_\delta \subseteq \bigcup \eta$. Let $a \stackrel{\text{def}}{=} a_{s^\delta(n)} \notin \eta$. Then $\operatorname{rk}^B(a \cdot (b_\delta - c_\delta)) < \operatorname{rk}^B(a)$. Also, $\operatorname{rk}^B(a - b_\delta) < \operatorname{rk}^B(a)$. Note that $a - c_\delta \leq a \cdot (b_\delta - c_\delta) + (a - b_\delta)$. So

$$\operatorname{rk}^B(a - c_\delta) \leq \max(\operatorname{rk}^B(a \cdot (b_\delta - c_\delta), \operatorname{rk}^B(a - b_\delta)) < \operatorname{rk}^B(a).$$

By the last claim of Part (b) of this theorem, and since $c_\delta \in I(B)$, there is a finite set $\sigma \subseteq \{\{n\}\colon n \in \omega\} \cup \{a_\alpha\colon \alpha \in \aleph_1\} \cup \{b_\delta\colon \delta \in \aleph_1^{\lim}\}$ such that $c_\delta \leq \bigcup \sigma$. So $b_\delta \preceq^B \bigcup \sigma$.

We apply (*) to σ. Hence there is $n \in \omega - F$ such that $a'_{s^\delta(n)} \not\subseteq^B \bigcup \sigma$. So

(i) $$a'_{s^\delta(n)} \not\subseteq^B c_\delta.$$

Since $n \notin F$, $a_{s^\delta(n)} \preceq^B c_\delta$. That is,

(ii) $$a_{s^\delta(n)} \precsim^{\text{fin}} c_\delta.$$

(i) and (ii) imply that

(iii) $$a_{s^\delta(n)} \sim^B c_\delta \cdot a_{s^\delta(n)} < a_{s^\delta(n)}.$$

Also,

(iv) $$c_\delta \cdot a_{s^\delta(n)} \in G.$$

(iii) and (iv) contradict the minimality of $a'_{s^\delta(n)}$. So B is not well-generated. ∎

4. On the stationary Knaster property

In this section we describe the iteration scheme which gives the required model to prove our consistency result. We shall bring first the notion "stationary Knaster" defined by Talayco in [Ta], and we reprove a result of A. Blass (in [Ta]) that this property is preserved under finite support iteration.

In a forcing notion $(P, <)$, $p < q$ means that q is more informative than p.

Definition 4.1: (a) A poset $\langle P, <\rangle$ satisfies the **stationary Knaster condition**, if for every stationary set $u \subseteq \aleph_1$, and a family $\{p_\alpha: \alpha \in u\} \subseteq P$, there is a stationary set $v \subseteq u$ such that for every $\alpha, \beta \in v$, p_α and p_β are compatible.

(b) A poset $\langle P, <\rangle$ satisfies the **weak version of the stationary Knaster condition**, if for every $\{p_\alpha: \alpha \in \aleph_1\} \subseteq P$, there is a stationary set $u \subseteq \aleph_1$ such that for every $\alpha, \beta \in u$, p_α and p_β are compatible.

THEOREM 4.2: *Let V be a universe of ZFC. Let $\langle \vec{a}, \vec{b}\rangle \in V$ be a stationarily special tail system. Let $\langle P, <\rangle \in V$ satisfy the stationary Knaster condition, and $G \subseteq P$ be V-generic. Then $\langle \vec{a}, \vec{b}\rangle$ is still stationarily special in $V[G]$.*

Proof: Let $\langle \vec{a}, \vec{b}\rangle$ be a stationarily special tail system, and suppose that it is no longer stationarily special in $V[G]$, where G is V-generic over a stationary Knaster poset P. This means (by the equivalent form of the definition) that there is a stationary set $w \in V[G]$ and a family $\{a'_\alpha: \alpha \in w\}$ that is near to $\vec{a}$ such that for every limit $\delta \in w$ the following holds. There is a finite set $\sigma = \sigma(\delta)$ such that $\sigma \subseteq \{a_\alpha: \alpha \in \aleph_1\} \cup \{b_\gamma: \gamma \in \aleph_1^{\lim}\} \cup \{\{n\}: n \in \omega\}$, and $\{n \in \omega: s^\delta(n) \in w \text{ and } a'_{s^\delta(n)} \not\subseteq \bigcup \sigma\}$ is finite.

Now we can find names for these objects and a condition q_0 forcing all of this to happen, and we want to derive a contradiction. Let $\widetilde{w}$ be a name of w and $\widetilde{a}'$ a name of $\{a'_\alpha: \alpha \in w\}$. So $\widetilde{a}'$ is a name of a function, and $\widetilde{a}'(\alpha)$ refers to the value of this function at α whenever $\alpha \in \aleph_1$ is in its domain. Let w^* be the set of all limit ordinals $\delta \in \aleph_1$ such that some $p \geq q_0$ forces that $\delta \in \widetilde{w}$. Clearly, $w^* \in V$, and since in $V[G]$ it is a superset of the stationary set of the limit ordinals in w, w^* is a stationary subset of $\aleph_1$ in V.

For every $\delta \in w^*$ let $p_\delta \geq q_0$ be a condition that forces the following relevant information on δ.

1. $p_\delta \Vdash \delta \in \widetilde{w}$.
2. For some $a'_\delta \subseteq \omega$ such that $a'_\delta \vartriangle a_\delta$ is finite, $p_\delta \Vdash a'_\delta = \widetilde{a}'(\delta)$.
3. For some $\sigma = \sigma(\delta) \subseteq \{a_\alpha: \alpha \in \aleph_1\} \cup \{b_\gamma: \gamma \in \aleph_1^{\lim}\} \cup \{\{n\}: n \in \omega\}$, and for some $n(\delta) \in \omega$, p_δ forces the following formula:

$$\{n \in \omega: s^\delta(n) \in \widetilde{w} \text{ and } \widetilde{a}'(s^\delta(n)) \not\subseteq \bigcup \sigma\} \text{ is contained in } n(\delta).$$

Now consider the sequence of conditions $\langle p_\delta : \delta \in w^* \rangle$ defined in V, and apply our assumption that P is a stationarily Knaster poset to get a stationary subset $u \subseteq w^*$ such that any two conditions with indices in u are compatible.

Then consider the sequence $\langle a'_\delta : \delta \in u \rangle$ defined in V, and apply our assumption that $\langle \vec{a}, \vec{b} \rangle$ is stationarily special to get an ordinal $\delta \in u$, δ limit, such that for any σ, and in particular for $\sigma = \sigma(\delta)$,

$$\{n \in \omega : s^\delta(n) \in u \text{ and } a'_{s^\delta(n)} \not\subseteq \bigcup \sigma\} \text{ is infinite.}$$

Now pick some $n > n(\delta)$ in this infinite set and consider the two conditions p_δ and $p_{s^\delta(n)}$. They are compatible, and yet they force incompatible information regarding $a'_{s^\delta(n)} \not\subseteq \bigcup \sigma$, a contradiction which proves the theorem. ∎

LEMMA 4.3: *Suppose that $(P, <)$ is a c.c.c. poset, $u \subseteq \aleph_1$ a stationary set, and $\{p_\alpha : \alpha \in u\}$ a family of conditions indexed by u. Let $\tilde{G}$ be the canonical name of the generic filter over P. Then there is a condition $p \in P$ such that*

$$p \Vdash^P \{\alpha \in u : p_\alpha \in \tilde{G}\} \text{ is stationary.}$$

Proof: We assume that our posets are **separative**. That is, if $q \not\leq p$, then there is $r \geq p$ such that r and q are incompatible. Thus $p \Vdash^P$ " $r \in \tilde{G}$ " iff $p \geq r$ in P. The lemma does not say that for some $p \in P$, $\{\alpha : p \geq p_\alpha\}$ is stationary, only that (p forces that) the set of those α's for which p_α is in the generic filter is stationary.

Recall that if G is a V-generic filter over the c.c.c. poset P, then every closed unbounded subset of $\aleph_1$ in $V[G]$ contains a closed unbounded subset which is in V. If there is no p as in the lemma, then there is a maximal antichain A in P (which is necessarily countable) such that for every $r \in A$ there is a closed unbounded set $C_r \subseteq \aleph_1$ such that

$$r \Vdash^P \forall \alpha \in u \cap \underset{\sim}{C_r},\ p_\alpha \notin \tilde{G}.$$

Define $C = \cap\{C_r : r \in A\}$. Then C is closed unbounded and so $C \cap u \neq \emptyset$. Let $\alpha \in C \cap u$, and consider p_α which must be compatible with some r in the maximal antichain A. Yet $p_\alpha \Vdash^P p_\alpha \in \tilde{G}$, and hence p_α is incompatible with r.

LEMMA 4.4: *Suppose that $\langle P, < \rangle$ satisfies the stationary Knaster condition, and $\tilde{Q}$ is a P-name of a poset such that*

$$0^P \Vdash^P \tilde{Q} \text{ has the stationary Knaster property.}$$

Then $P * \tilde{Q}$ has the stationary Knaster property.

Proof: Let $u \subseteq \aleph_1$ be a stationary set and $\vec{a} = \{\langle p_\alpha, \tilde{q}_\alpha \rangle : \alpha \in u\} \subseteq P * \tilde{Q}$. We must find a stationary subset $u_0 \subseteq u$ such that any two conditions with indices in u_0 are compatible.

Let $\tilde{w}$ be a canonical name such that whenever G is a V-generic filter over P, then $\tilde{w}$ is interpreted as $w = \{\alpha \in u : p_\alpha \in G\}$. By the previous lemma, there is $p \in P$ such that $p \Vdash \tilde{w}$ is stationary.

Now, since 0^P forces that $\tilde{Q}$ has the stationary Knaster property, there is a P-name $\tilde{v}$ such that

$$p \Vdash^P (\tilde{v} \subseteq \tilde{w}) \text{ is stationary}) \wedge$$
$$(\forall \alpha, \beta \in \tilde{v})(\tilde{q}_\alpha \text{ and } \tilde{q}_\beta \text{ are compatible in } \tilde{Q}).$$

Let $v = \{\alpha : (\exists r \in P)((r \geq p) \wedge (r \Vdash^P \alpha \in \tilde{v})\}$.

Clearly v is stationary. For every $\alpha \in v$ let $r_\alpha \geq p$ be such that $r_\alpha \Vdash^P \alpha \in \tilde{v}$. So $r_\alpha \Vdash^P \alpha \in \tilde{w}$, and hence $r_\alpha \Vdash^P \underset{\sim}{p_\alpha} \in \tilde{G}$. As P is separative, $r_\alpha \geq p_\alpha$.

Since P has the stationary Knaster property, there is a stationary set $v_0 \subseteq v$ such that for every $\alpha, \beta \in v_0$, r_α and r_β are compatible.

We show that for every $\alpha, \beta \in v_0$, $\langle p_\alpha, \tilde{q}_\alpha \rangle$ and $\langle p_\beta, \tilde{q}_\beta \rangle$ are compatible. For every $\alpha, \beta \in v_0$ take $r \geq r_\alpha, r_\beta$. Then $r \Vdash^P \alpha, \beta \in \tilde{v}$. Hence $r \Vdash^P \tilde{q}_\alpha$ and $\tilde{q}_\beta$ are compatible in $\tilde{Q}$. Thus $\langle p_\alpha, \tilde{q}_\alpha \rangle$ and $\langle p_\beta, \tilde{q}_\beta \rangle$ are compatible. So the lemma is proved ∎

We now have the following corollary.

LEMMA 4.5: *Suppose that $\langle P, \leq \rangle$ is a poset obtained from a finite support iteration of posets with the stationary Knaster property. That is, there is a finite support system $\langle \{P_\alpha : \alpha \leq \delta\}, \{\tilde{Q}_\alpha : \alpha < \delta\} \rangle$ such that $P = P_\delta$ and for every $\alpha < \delta$, $0^{P_\alpha} \Vdash^{P_\alpha}$ "$\tilde{Q}_\alpha$ has the stationary Knaster property". Then $\langle P, \leq \rangle$ has the the stationary Knaster property.*

Proof: Although the pattern of the proof is not different from similar preservation proofs, we give a detailed argument for the sake of complexness. We may assume that P is constructed in the following way.

(1) For every $\alpha < \delta$:

(i) $\langle P_\alpha, \leq^{P_\alpha} \rangle$ is a poset with a minimum 0^{P_α},

(ii) $\tilde{Q}_\alpha$, $\leq^{\tilde{Q}_\alpha}$ and $0^{\tilde{Q}_\alpha}$ are P_α-names, and $0^{\tilde{Q}_\alpha} \in \text{Rng}(\tilde{Q}_\alpha)$, and

(iii) $0^{P_\alpha} \Vdash^{P_\alpha}$ "$\langle \tilde{Q}_\alpha, \leq^{\tilde{Q}_\alpha} \rangle$ is a poset, and $0^{\tilde{Q}_\alpha} = \min(\tilde{Q}_\alpha)$".

(2) For every $\alpha \le \delta$, and f: $f \in P_\alpha$ iff

(i) f is a function and $\mathrm{Dom}(f) = \alpha$,

(ii) for every $\beta < \alpha$, $f{\restriction}\beta \in P_\beta$, $f(\beta) \in \mathrm{Rng}(\tilde{Q}_\beta)$ and $f{\restriction}\beta \Vdash^{P_\beta} f(\beta) \in \tilde{Q}_\beta$,

(iii) $\{\beta < \alpha: f(\beta) \ne 0^{\tilde{Q}_\beta}\}$ is finite.

(3) (i) For every $\alpha < \delta$ and $f, g \in P_{\alpha+1}$, $f \le^{P_{\alpha+1}} g$ iff $f{\restriction}\alpha \le^{P_\alpha} g{\restriction}\alpha$ and $g{\restriction}\alpha \Vdash^{\tilde{Q}_\alpha} f(\alpha) \le^{P_\alpha} g(\alpha)$.

(ii) For every limit ordinal $\alpha \le \delta$ and $f, g \in P_\alpha$, $f \le^{P_\alpha} g$ iff for every $\beta < \alpha$, $f{\restriction}\beta \le^{P_\beta} g{\restriction}\beta$.

For $f \in P_\delta$ let $\mathrm{supp}(f) = \{\alpha < \delta: f(\alpha) \ne 0^{\tilde{Q}_\alpha}\}$. We observe the following facts.

(*) For every $\alpha < \delta$ the function from $P_{\alpha+1}$ to $P_\alpha * \tilde{Q}_\alpha$ defined by $f \mapsto \langle f{\restriction}\alpha,\ f(\alpha)\rangle$ is an isomorphism between $P_{\alpha+1}$ and $P_\alpha * \tilde{Q}_\alpha$.

(**) For every $\beta < \delta$, $\langle\{P_\alpha: \alpha \le \beta\},\ \{\tilde{Q}_\alpha: \alpha < \beta\}\rangle$ is a finite support system. Also, let $P'_\beta = \{f \in P_\delta: \mathrm{supp}(f) \subseteq \beta\}$; then $\langle P'_\beta,\ \le^{P_\delta} {\restriction} P'_\beta\rangle \cong \langle P_\beta,\ \le^{P_\beta}\rangle$.

(* * *) Let $f, g \in P_\delta$ and $\alpha < \delta$ be such that: (i) $f{\restriction}\alpha$ and $g{\restriction}\alpha$ are compatible in P_α, and (ii) $\mathrm{supp}(f) \cap \mathrm{supp}(g) \subseteq \alpha$. Then f and g are compatible in P_δ.

We prove the claim of the lemma by induction on δ.

Suppose that $\delta = \alpha + 1$. Then by the induction hypothesis, P_α has the stationary Knaster property, and then by (*) and the previous lemma, $P_{\alpha+1}$ has the stationary Knaster property.

Suppose that $\mathrm{cf}(\delta) = \aleph_0$. Let $u \subseteq \aleph_1$ be a stationary set, and $\{p_\alpha: \alpha \in u\} \subseteq P_\delta$. Let $\{\beta_n: n \in \omega\}$ be a strictly increasing sequence converging to δ. There is $n \in \omega$ such that $u_n \stackrel{\text{def}}{=} \{\alpha \in u: \mathrm{supp}(p_\alpha) \subseteq \beta_n\}$ is stationary. By the induction hypothesis applied to β_n and by (**), there is a stationary set $v \subseteq u_n$ such that $\{p_\alpha{\restriction}\beta_n: \alpha \in v\}$ is a pairwise compatible family in P_{β_n}. So obviously, $\{p_\alpha: \alpha \in v\}$ is a pairwise compatible family in P_δ. Hence P_δ has the stationary Knaster property.

Suppose that $\mathrm{cf}(\delta) > \aleph_1$. Let $u \subseteq \aleph_1$ be a stationary set, and $\{p_\alpha: \alpha \in u\} \subseteq P_\delta$. Then there is $\beta < \delta$ such that for every $\alpha \in u$, $\mathrm{supp}(p_\alpha) \subseteq \beta$. So by (*) and the induction hypothesis, there is a stationary set $v \subseteq u$ such that $\{p_\alpha{\restriction}\beta: \alpha \in v\}$ is a pairwise compatible family in P_β. So obviously, $\{p_\alpha: \alpha \in v\}$ is a pairwise compatible family in P_δ. Hence P_δ has the stationary Knaster property.

Suppose that $\mathrm{cf}(\delta) = \aleph_1$. Let $u \subseteq \aleph_1$ be a stationary set, and $\{p_\alpha: \alpha \in u\} \subseteq P_\delta$. Let $\{\eta_i: i \in \aleph_1\}$ be a strictly increasing continuous sequence converging to δ such that $\eta_0 = 0$. For every $\alpha \in u$, let $\sigma_\alpha = \{i \in \aleph_1: \mathrm{supp}(p_\alpha) \cap [\eta_i, \eta_{i+1}) \ne \emptyset\}$. So σ_α is finite. Let $v \subseteq u$ be a stationary set such that $\{\sigma_\alpha: \alpha \in v\}$ is a Δ-system.

Let $j < \aleph_1$ be such that for every distinct $\alpha, \beta \in v$, $\sigma_\alpha \cap \sigma_\beta \subseteq j$. By the induction hypothesis applied to η_j, there is a stationary set $w \subseteq v$ such that for every $\alpha, \beta \in w$, $p_\alpha \restriction P_{\eta_j}$ and $p_\beta \restriction P_{\eta_j}$ are compatible. Clearly, for every $\alpha, \beta \in w$, $\mathrm{supp}\,(p_\alpha) \cap \mathrm{supp}\,(p_\beta) \subseteq \eta_j$. So by $(***)$, p_α and q_α are compatible. ∎

For $f, g \in \omega^\omega$ let $f \prec^{\mathrm{fin}} g$ mean that there is $n \in \omega$ such that for every $k > n$, $f(k) < g(k)$. Let λ be a regular cardinal. We say that ω^ω has a strong cofinality λ, if $\langle \omega^\omega,\ \prec^{\mathrm{fin}} \rangle$ has a cofinal chain of type λ.

It is obvious that if ω^ω has a strong cofinality λ, then the unboundedness number of ω^ω is λ.

The following forcing set appears in Hechler [H]. Let $\langle P_D,\ \leq^{P_D} \rangle$ be the poset that adds a function $f\colon \omega \to \omega$ such that f dominates all members of $(\omega^\omega)^V$. So a member p of P_D has the form $\langle g^p,\ \sigma^p \rangle$, where for some $n^p \in \omega$, $g^p\colon [0, n^p) \to \omega$, and σ^p is a finite subset of ω^ω. The partial ordering of P_D is defined as follows: $p \leq^{P_D} q$ if

(i) $g^p \subseteq g^q$;

(ii) $\sigma^p \subseteq \sigma^q$; and

(iii) for every $h \in \sigma^p$ and $k \in [n^p, \omega) \cap \mathrm{Dom}\,(g^q)$, $h(k) < g^q(k)$.

If R is a poset, let $\underset{\sim}{Q}_D(R)$ denote an R-name such that for every V-generic filter $G \subseteq R$, $\mathrm{val}\,(\underset{\sim}{Q}_D(R), G) = (P_D)^{V[G]}$.

Let λ be a cardinal, and let $\langle \{P^D_\alpha\colon \alpha \leq \lambda\},\ \{\underset{\sim}{Q}_D(P^D_\alpha)\colon \alpha < \lambda\} \rangle$ be a finite support system.

Part (b) of the next lemma appears in Hechler [H].

LEMMA 4.6: *Suppose that λ is a regular cardinal.*

(a) *P^D_λ has the stationary Knaster property.*

(b) *Let $G \subseteq P^D_\lambda$ be a generic filter. Then in $V[G]$, ω^ω has strong cofinality λ.*

Proof: (a) Obviously, P_D is σ-centered, so it has the stationary Knaster property. By Lemma 4.5, P^D_λ has the stationary Knaster property.

(b) The claim of (b) is easy and well-known. See [H].

Proof of Theorem 1.1: Assume $V = L$, and then $\diamondsuit^*_{\aleph_1}$ holds. By Theorem 2.3 there is a special tail system $\langle \vec{a},\ \vec{b} \rangle \in V$. By the list of implications (†), appearing after Definition 2.2, $\langle \vec{a},\ \vec{b} \rangle$ is also a stationarily special tail system.

Let G be a $P^D_{\aleph_2}$-generic filter over V and $W = V[G]$. By Lemma 4.6(a) and Theorem 4.2,

$$W \models \langle \vec{a},\ \vec{b} \rangle \text{ is a stationarily special tail system.}$$

By (†) again, $\langle \vec{a}, \vec{b} \rangle$ is a weakly special tail system in W. Let B be the Boolean algebra of $\langle \vec{a}, \vec{b} \rangle$. Then by Theorem 3.3(a), the cardinal sequence of B is $\langle \aleph_0, \aleph_1, \aleph_1, 1 \rangle$, and by Theorem 3.3(c), B is not well-generated.

By Lemma 4.6(b), $\mathrm{b}^{(W)} = \aleph_2$. So the theorem is proved. ∎

Note that in the above proof, we may replace $\aleph_2$ by any regular cardinal.

There is another route which leads to the conclusion of Theorem 1.1. We can replace Theorem 4.2 by the following theorem.

THEOREM 4.7: *Let V be a universe of ZFC. Let $\langle \vec{a}, \vec{b} \rangle \in V$ be a stationarily special tail system. Let $\langle P, < \rangle \in V$ satisfy the weak stationary Knaster condition, and $G \subseteq P$ be V-generic. Then $\langle \vec{a}, \vec{b} \rangle$ is weakly special in $V[G]$.*

Remark: Here, we start with a tail system $\langle \vec{a}, \vec{b} \rangle$ whose specialness property is stronger than in Theorem 4.2. The specialness property of $\langle \vec{a}, \vec{b} \rangle$ after the forcing is weaker than the one obtained in Theorem 4.2. However, here, P is assumed to have only the weak stationary Knaster property, whereas in Theorem 4.2 it was assumed to have the stationary Knaster property.

Proof: Let $\langle \vec{a}, \vec{b} \rangle$ and $\langle P, < \rangle$ be as in the theorem, and suppose by contradiction that $\langle \vec{a}, \vec{b} \rangle$ is not weakly special in $V[G]$. We may assume that

$$0^P \Vdash \langle \vec{a}, \vec{b} \rangle \text{ is not weakly special.}$$

Let τ be a P-name such that 0^P forces that τ is near to $\vec{a}$, and that τ is a counter-example to the fact that $\langle \vec{a}, \vec{b} \rangle$ is weakly special. Let $\tau(\alpha)$ denote a name of the α's element of τ. That is, $0^P \Vdash$ "$\tau(\alpha)$ is the α's element of τ".

For every $\delta \in \aleph_1^{\lim}$ let $p_\delta \in P$ and $a'_\delta, c_\delta \subseteq \omega$ satisfy the following:

(1) $p_\delta \Vdash \tau(\delta) = a'_\delta$ (hence $a'_\delta \sim^{\text{fin}} a_\delta$);

(2) there is a finite set $\sigma \subseteq \{\{n\}: n \in \omega\} \cup \{a_\alpha: \alpha \in \aleph_1\} \cup \{b_\delta: \delta \in \aleph_1^{\lim}\}$ such that $c_\delta = \bigcup \sigma$;

(3) $p_\delta \Vdash \bigcup_{n\in\omega} \tau(s^\delta(n)) \subseteq c_\delta$.

Since P has the weak stationary Knaster property, there is a stationary set $u \subseteq \aleph_1^{\lim}$ such that for every $\alpha, \beta \in u$, p_α and p_β are compatible.

Let $\vec{a}' = \{a'_\alpha: \alpha \in u\}$. Since u is stationary, and $\langle \vec{a}, \vec{b} \rangle$ is strongly stationarily special, there is $\delta \in u^{\lim} \cap u$ such that (i) and (ii) of Definition 2.2(b) hold for $\vec{a}'$.

By 2.2(b)(i), $\{n \in \omega: s^\delta(n) \in u\}$ is infinite.

By 2.2(b)(ii), $\{n \in \omega: s^\delta(n) \in u \text{ and } a'_{s^\delta(n)} \subseteq c_\delta\}$ is finite.

So there is n such that $\alpha \stackrel{\text{def}}{=} s^\delta(n) \in u$ and $a'_\alpha \not\subseteq c_\delta$.

Since $\alpha, \delta \in u$, p_α and p_δ are compatible. So let $q > p_\alpha, p_\delta$. Then $q \Vdash \tau(\alpha) = a'_\alpha$. So $q \Vdash \bigcup_{n\in\omega} \tau(s^\delta(n)) \not\subseteq c_\delta$. This contradicts the fact that $p_\delta \Vdash \bigcup_{n\in\omega} \tau(s^\delta(n)) \subseteq c_\delta$. This proves the theorem. ∎

The proof of Theorem 1.1 remains the same.

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